Sheet 0

These are meant for revision, not for handing in. But *do ask class tutor* if any are problematic!

R is a commutative ring with 1, I and J are ideals of R. F is a field.

PID = Principal ideal domain: an integral domain R in which every ideal is *principal*, i.e. of the form aR, $a \in R$.

1. (i) Verify that $I + J$, $I \cap J$ and IJ are ideals, and that $IJ \subseteq I \cap J$.

(ii) Suppose that $I + J = R$. Show that $I^n + J^m = R$ for all $m,\ n \in \mathbb{N}$. (I^n denotes $I \cdot I \cdot \ldots\ldots \cdot I$ with n factors). Show that $IJ = I \cap J$. Show that

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}.$$

(This should remind you of the *Chinese Remainder Theorem*.) What is the identity element of this direct product of rings?

2. Show that I is a maximal ideal if and only if R/I is a field ('maximal' means maximal w.r.t. inclusion in the set of all proper ideals). Deduce that maximal ideals are prime.

3. Let P be a prime ideal of R. Show that if $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

4. Which of the following rings are PIDs? (i) $\mathbb{Z}[t]$ (polynomial ring in one variable); (ii) $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$; (iii) $\mathbb{Z}[\sqrt{-1}]$; (iv) $F[t]$; (v) $F[t_1, t_2]$. (*Ans.: just* (iii) *and* (iv). *Prove this*!)

5. Suppose that $I = aR$ is principal. (i) Show that $I = R$ iff a is a *unit* (invertible element of R).

Now assume that $R \neq I \neq 0$. (ii) Prove that I is prime if and only if a is a *prime element* of R (i.e. if a divides bc then a divides b or a divides c.)

Assume further that R is an integral domain. (iii) Show that if aR is prime and

$$aR \subseteq bR \neq R$$

then $aR = bR$. Deduce that if R is a PID then every prime ideal is maximal.

(iv) Prove that if I is maximal then a is an *irreducible element* (i.e. if $a = bc$ then b is a unit or c is a unit).

(v) Prove the converse of (iv) in the case where R is a PID. Deduce that *every PID is a UFD* (unique factorization domain).

(vi) Let $R = \mathbb{Z}[\sqrt{-5}]$; show that 2 is irreducible in R but $2R$ is not a maximal ideal (*Hint*: verify that $2R \subset 2R + (1 + \sqrt{-5})R \subset R$.)